Chapter 12

The Precision and Internal Confidence of Our Approximate Number Thoughts

Justin Halberda and Darko Odic

Department of Psychological and Brain Sciences, Johns Hopkins University, Baltimore, MD, USA

INTRODUCTION

We all have an approximate sense for numbers. That is, we have experiences of estimating of how many voices we hear or stars we see, and we have experiences of ordinal relation such as judging that the number of voices we hear right now is fewer than the number of stars we see in the sky right now. The sense of number that supports these kinds of experiences is quite immediate (in perception) and primitive (i.e., operating from birth and serving as a foundation on which we learn about the world as understood through numbers). We share this primitive sense of number with other animals (i.e., it has been measured in nonhuman primates, mammals more broadly, birds and fish; e.g., Cantlon & Brannon, 2006; Hauser, Tsao, Garcia, & Spelke, 2003; Meck & Church, 1983; Nieder & Miller, 2004), with babies from birth (Izard, Sann, Spelke, & Streri, 2009), with individuals from every human culture (NB, even those with no written mathematics of any kind; e.g., Dehaene, Izard, Spelke, & Pica, 2008; Frank, Everett, Fedorenko, & Gibson, 2008; Gordon, 2004; Spaepen, Coppola, Spelke, Carey, & Goldin-Meadow, 2011), and with every human-like mind that has ever walked the face of this earth (e.g., the painters of the Chauvet cave and Jesus of Nazareth).[1]

The portion of cognition that generates these experiences of approximate number has been called an "approximate number system" (ANS). It is

1. But note that homology has yet to be demonstrated—and, a prudent theorist might bet that the number sense is a case of convergent evolution in fish and humans or insects and humans given that we know of so little at the system's neuroscience scale that would currently count as homologous among these organisms (special thanks to Alvaro Mailhos and Dave Geary for discussions and inspiration surrounding this point).

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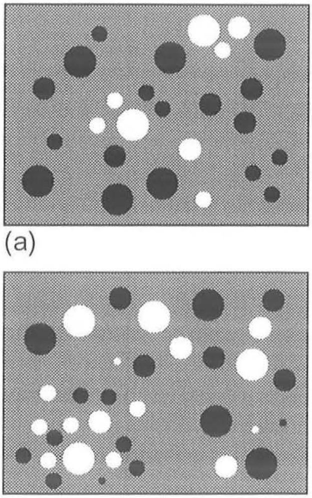
generated by neurons in the intraparietal sulcus (IPS) and area lateral intraparietal (LIP) cortex (Nieder, 2005; Nieder & Miller, 2004; Piazza, Izard, Pinel, Le Bihan, & Dehaene, 2004; Roitman, Brannon, & Platt, 2007). It is an evolutionarily ancient and primitive system for numerical thought, and yet, surprisingly, individual differences in the accuracy and precision of these approximate number representations relate to our performance in symbolic, formal, school mathematics (for review, Feigenson, Libertus, & Halberda, 2013; but see De Smedt, Noël, Gilmore, & Ansari, 2013).

Much of the evidence that supports our current understanding of the ANS is reviewed in other chapters within this volume (e.g., Cantlon, this volume; Starr & Brannon, this volume; vanMarle, this volume), so we try not to duplicate those here. Instead, our aim is to describe the psychophysical model describing ANS representations and their precision (i.e., the Weber fraction and related concepts) and to suggest a reason why this precision may differ across observers. We've organized the chapter into three major sections. We begin by discussing the critical ANS behavioral signatures, including internal confidence, individual differences, and ratio dependence. Subsequently, we review the psychophysical model that accounts for these signatures, including a discussion on the nature of the Weber fraction (w), which we conceptualize as a scaling factor that determines the precision of all ANS representations. Finally, we elaborate on this model and argue that the ANS's key role is in providing "internal confidence" to the observer, rather than the absolute number of items in a scene.

BEHAVIORAL AND NEURAL SIGNATURES OF THE ANS

The ANS has been extensively studied, and there are many well-established signatures of its use (see following sections). Here, we focus on three behavioral signatures that we believe are central to the proper understanding of the ANS: the internal confidence generated by decisions about approximate number; individual differences in ANS performance; and the effect of numerical ratio on accuracy, response time, and internal confidence.

The heart of the ANS (and the psychological experiences of number that it generates) is its ordinal and approximate character. Consider the images in Figure 12-1 (NB, the ANS is multimodal, e.g., Nieder, 2012, and provides a sense of number for both voices heard and stars seen, but our examples focus on vision for ease of demonstration; Baker & Jordan, this volume). The reader will likely find it quite simple to judge that there are more black dots than white dots in Figure 12-1a, even with only a brief glance. It is also likely that deciding whether there are more black dots than white dots in Figure 12-1b is a bit more difficult. Irrespective of the reader's answer to this particular "more" question, we invite you to reflect on your internal confidence for any ordinal guess you might make with respect to Figures 12-1a and 12-1b. For which figure, 12-1a or 12-1b, would you feel more confident about your



(a)

(b)

FIGURE 12-1 Quick. Are more of the dots black or white? You should find (a) to be easier, quicker, and that your internal confidence in your decision should be higher.

answer, after a brief glance? Which figure are you more likely to make a mistake on?

We expect that the reader feels more confident about his or her guess for Figure 12-1a (i.e., "Black has more!") than for Figure 12-1b (i.e., "Black has more???"). Experiments in multiple labs have found that this difference in internal confidence (and a corresponding increase in errors and response time as trials become more difficult) is not simply a result of there being more dots in Figure 12-1b than 12-1a, nor of missing a few dots or counting a dot twice by accident (Cordes, Gallistel, Gelman, & Latham, 2007; Cordes, Gelman, Gallistel, & Whalen, 2001). As expanded on in the following sections, we believe that a sense of internal confidence for number thoughts (e.g., Figure 12-1a resulting in higher confidence than Figure 12-1b) is the most important psychological experience that the ANS gives rise to—rather than experiences of any particular cardinality (e.g., "that looks like around 18 dots")—and that the ANS and all other magnitude dimensions (e.g., time, length, loudness, brightness; Lourenco, this volume) are in the business of supporting ordinal comparisons and not absolute judgments (i.e., more-black-dots, and not approximately-7-black-dots*).[2]

2. *A fortiori*, estimation (e.g., "that looks like around 16 dots") is a case of relative, ordinal comparison (see also Laming, 1997) and probably always involves comparisons to internal standard comparators (see, e.g., Izard & Dehaene, 2008; Sullivan & Barner, 2013). A more thorough defense of this view is beyond the scope of this chapter, but we hope that work in progress, as well as a book in progress, will be able to explore these issues in greater detail over the coming years.

One important lesson that can be drawn from viewing Figures 12-1a and 12-1b is that the internal confidence you feel when viewing them arises incredibly rapidly, probably even before you have determined your answer to the number question (e.g., "Hmm. . .Black has more!"). As a result, internal confidence, or lack thereof, is felt throughout the "Hmm. . ." period leading up to your decision, and informs and guides your evidence-gathering procedures (e.g., adjusting the parameters of an internal size normalization algorithm, inspecting with greater scrutiny the visually crowded regions of the image). For example, prolonged exposure to low confidence trials impairs subsequent discrimination performance, whereas exposure to high confidence trials improves it (Odic, Hock, & Halberda, 2012; Wang, Odic, Halberda, & Feigenson, under review). Much of the recent work in our lab demonstrates that internal confidence is continuous and quantitatively rich (i.e., not merely "high, low, or medium"), and may be the primary source for the individual differences in numerical estimation and discrimination performance in tasks that measure our abilities to estimate and compare numerosities (Odic et al., 2012; Wang et al., under review). This sense of internal confidence has yet to be fully explored empirically, and this chapter serves as an introduction to ideas that are currently in development.

The second important lesson to draw from viewing Figures 12-1a and 12-1b is that different observers will experience different levels of internal confidence for these same numerical judgments. If you happen to have children nearby as you read this, we encourage you to ask them to give you their opinion about Figures 12-1a and 12-1b—whether there are more black dots or white dots in each figure. You should find that they take longer to answer than you would take (try to convince them to answer without explicit verbal counting), but that they, like you, answer more quickly for Figure 12-1a than Figure 12-1b.

Individual and developmental differences are also found in ANS discrimination: individuals vary widely in both the speed and accuracy of deciding whether more of the dots are black or white, and ANS speed and accuracy gradually improve over development (Halberda & Feigenson, 2008; Halberda, Ly, Wilmer, Naiman, & Germine, 2012; Odic, Libertus, Feigenson, & Halberda, in press; Piazza et al., 2010). These individual and developmental differences are usually measured through Weber fractions (w), a key concept describing the acuity or precision of an individual's ANS. For example, w has been found to be impaired in individuals who struggle with dyscalculia, or math learning disability (MLD; Mazzocco, Feigenson, & Halberda, 2011; Piazza et al., 2010). In what follows, we discuss the nature of the Weber fraction and the ANS psychophysical model, and subsequently return to unifying it with the concept of internal confidence.

The third important behavioral signature of note is that children and adults should also answer faster and provide more accurate responses for Figure 12-1a than Figure 12-1b. The faster and more accurate answering for Figure 12-1a compared to 12-1b is a universal behavioral signature that every organism shows when relying on approximate number representations (i.e., you, us, children, rats, pigeons). All creatures so far tested have shown this kind of ratio-dependent responding (i.e., Weber's law), where we are both faster and more accurate for "easier" numerical comparisons, and our performance degrades as the ratio between the two numbers being compared moves closer to 1 (e.g., the numerosities in Figure 12-1b are closer to a ratio of 1 where the two sets of dots would be equal in number and there would, therefore, be no correct answer; Figure 12-1a ratio = 18/8 = 2.25; Figure 12-1b ratio = 18/16 = 1.125).

Internal confidence, individual differences, and ratio dependence are three signatures that are of great value for promoting our understanding of the approximate number system. Research scientists are relying on these patterns to help inform their understanding of the functioning of the ANS. All formal models of ANS representation must strive to provide a detailed account for why these patterns have been observed in every study of ANS performance yet published and in every animal yet tested. Why do these patterns emerge? Like the flow of sunspots across the face of the sun was for Galileo (i.e., and the systematic relationship between the time of year and the angle of their traversal), the elegant systematicity of these response time and error distributions in numerical tasks is a coded key whose proper description will help unlock a door unto our more accurate understanding of numerical cognition and the foundations of our numerical thoughts.

In the remaining sections of this chapter, we review a psychophysical model that accounts for ratio-dependence, internal confidence, and individual and developmental differences in the ANS. But researchers have also identified numerous other signatures of interest; in the end, a sufficient theory of our approximate number representations should be able to provide explanations for all the observed patterns in the data, and not just a subset of them. Due to space constraints, we can only briefly review the list of other relevant signatures:

- Longer response times and higher error rates for younger observers and for observers who struggle with a math learning disability; each discussed in connection with Figure 12-1.
- Multimodal representations of number that include vision, audition, tactation, as well as serial and parallel presentation of collections through these modalities; e.g., the "voices heard and stars seen" mentioned previously (Izard et al., 2009; Lipton & Spelke, 2003; Nieder, 2012). Similarly, cue-combination effects of combining evidence across two or more modalities; e.g., the increased confidence we feel in our estimate of the number of people who are around us when we can both hear and see them talking around the campfire (Jordan & Brannon, 2006; Raposo, Sheppard, Schrater, & Churchland, 2012).
- Relationships between physical parameters of stimulus presentation and a resulting sense for numerosity, including dimensions such as visual

density (Dakin, Tibber, Greenwood, Kingdom, & Morgan, 2011; Durgin, 1995) and physically intermixing or separating sets of stimuli (Gebuis & Reynvoet, 2012; Price, Palmer, Battista, & Ansari, 2012; Zosh, Halberda, & Feigenson, 2011); that is, how do our sensory systems take perceptual evidence and translate this evidence into a numerical thought?

- Relations across various psychological dimensions that may share representational resources or formats with approximate number, such as temporal duration or surface area (Hurewitz, Gelman, & Schnitzer, 2006; Odic et al., in press), as well as effects of forming mappings across these various dimensions, such as longer lines or tone durations mapping naturally to larger numerosities, including the SNARC (spatial–numerical association of response codes) effect wherein observers tend to associate physical space from left to right with a mental number line ordered from small numbers on the left to larger numbers on the right (Gevers, Verguts, Reynvoet, Caessens, & Fias, 2006; Wood, Willmes, Nuerk, & Fischer, 2008),
- Effects of memory and executive control for approximate number representation and adjusting to the varying contexts of different display parameters (Gilmore et al., 2013; Pailian, Libertus, Feigenson, & Halberda, under review).
- The relation between perceptual effects of numerosity and later cognitive effects of numerosity, such as the effects of clustering or Gestalt grouping in a visual display (Im, Zhong, & Halberda, 2013); visual adaptation effects (Burr & Ross, 2008; Ross & Burr, 2010); visual and auditory parsing of individual items that form a collection (Franconeri, Bemis, & Alvarez, 2009; Halberda, Sires, & Feigenson, 2006).
- Mappings between nonverbal approximate number representations and formal math words and symbols; e.g., that "around ten" can be an approximation (Barth, Starr, & Sullivan, 2009; Le Corre & Carey, 2007; Mundy & Gilmore, 2009; Odic, Le Corre, & Halberda, under review; Sullivan & Barner, in press), and mappings between approximate number representations and spatial understandings of a number line; e.g., that "$6 > 5$" picks out an ordinal direction on the mental number line (Booth & Siegler, 2006; Opfer & Siegler, 2007; Siegler & Booth, 2004).
- Neuropsychological evidence, such as evidence for various deficits that emerge from brain damage, that inform which brain regions support our ANS representation and what other psychological functions may be supported by those regions (Dehaene & Cohen, 1997), as well as imaging studies of the human brain that also help to address these questions; e.g., such studies suggest that overlapping brain regions support our representations of approximate numbers and approximate areas (Castelli, Glaser, & Butterworth, 2006; Pinel, Piazza, Le Bihan, & Dehaene, 2004).
- Neurophysiological recordings that provide data to fuel our theorizing about the representational format and implementations code for numerical

representations as well as information about how such codes may differ across various brain regions (e.g., IPS versus LIP; Nieder, 2005; Piazza et al., 2004; Roitman et al., 2007).

All of these sources of evidence are important and should eventually inform theories of approximate number representation.

A PSYCHOPHYSICAL MODEL FOR ANS REPRESENTATIONS

The key to understanding ratio dependence, individual and developmental differences, and internal confidence (as well as the other signatures described previously) is understanding the psychophysical model of the ANS and correctly understanding what a Weber fraction is.

When we just glance at a picture, even without an explicit task, our experience of Figure 12-1a feels inherently comparative (e.g., "there are more black dots!"); that is, it would be very surprising if someone glanced at Figure 12-1a and reported, "Well, I see one specific dot on the bottom right" (implying that "I see nothing else on the page worth reporting") or "I see approximately 18 black dots and nothing else worthy of note." Displays like Figures 12-1a and 12-1b have been used to measure human and animal numerical discrimination performance (i.e., how accurate we are at determining which color has more dots after just a quick glance); such tasks are called "discrimination tasks."

To model our accuracy (and internal confidence) for judgments that engage the approximate number system (i.e., the "more" judgments we made for Figures 12-1a and 12-1b), we must first specify a model for the underlying ANS representations. It is generally agreed that our internal response to a numerosity in the world is a distribution of activation on a mental "number line." These distributions are inherently variable (sometimes called "noisy") and do not represent number exactly or discretely (Dehaene, 1997; Gallistel & Gelman, 2000). This means that there is some error each time they represent number, and this error can be thought of as a spread of activation around the number being represented. The mental number line is often modeled as having linearly increasing means and linearly increasing standard deviations (Gallistel & Gelman, 2000).[3] In such a format, the representation for, e.g., approximately-7 is a normal (Gaussian) probability density function that has its mean at 7 on the mental number line and a smooth degradation to either side of approximately-7; hence, approximately-6 and approximately-8

3. The mental number line has also been conceived of as logarithmically organized with constant standard deviation (Dehaene, 2003). Either this format or the linear one in Figure 12-2a results in the ratio-dependent performance that is the hallmark of the ANS. We rely on the linear format, as it generates fairly intuitive graphs (e.g., Cordes et al., 2001; Gallistel & Gelman, 2000; Meck & Church, 1983, Whalen et al., 1999).

on the mental number line are also highly activated by instances of sevenness in the world.

In Figure 12-2a, we have drawn curves that depict the ANS representations for numerosities 4–10. You can think of these curves as representing the location and spread of activity generated on a mental number line by a particular collection of items in the world with a different bump for each numerosity you might experience (e.g., 4, 5, or 6 black dots). Rather than activating a single discrete value (e.g., 7), the curves are meant to indicate that a range of activity is present each time a collection of (e.g., 7) items is presented.

In fact, the bell-shaped, or Gaussian, ANS representations depicted in Figure 12-2a are more than just a theoretical fantasy; "bumps" like these have been observed in neuronal recordings of the cortex of awake behaving monkeys as they engage in numerical discrimination tasks (e.g., shown an array of 7 dots, neurons that are preferentially tuned to representing approximately-7 are most highly activated, while neurons tuned to approximately-6 and approximately-8 are also fairly active, and those tuned

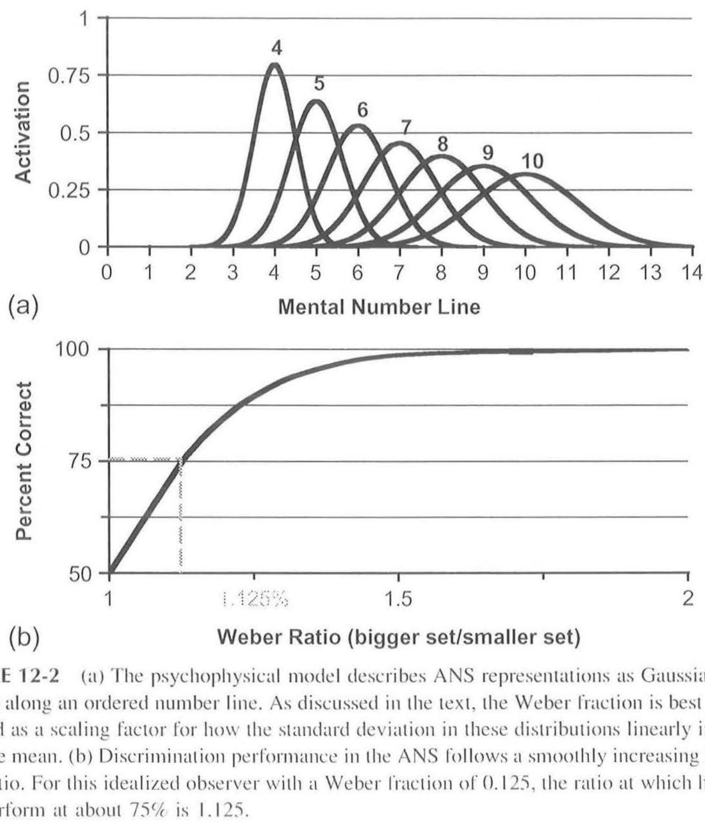


FIGURE 12-2 (a) The psychophysical model describes ANS representations as Gaussian distributions along an ordered number line. As discussed in the text, the Weber fraction is best conceptualized as a scaling factor for how the standard deviation in these distributions linearly increases with the mean. (b) Discrimination performance in the ANS follows a smoothly increasing function with ratio. For this idealized observer with a Weber fraction of 0.125, the ratio at which he or she will perform at about 75% is 1.125.

to approximately-5 and approximately-9 are active only slightly above their resting state; Nieder, 2005; Nieder & Miller, 2004). These neurons are found in the monkey brain in roughly the same region of cortex that has been found to support approximate number representations in human subjects in fMRI studies (Piazza et al., 2004).

It is important to keep in mind that this type of spreading activation is common throughout the cortex, and it is not unique to ANS representations. For example, we (and a rat) will also have neurons in our hippocampi that are preferentially tuned to particular locations in our office/bedroom/cramped-but-well-ventilated-cage that represent our position in space as we move around, with a spreading activation quite similar to the spreading activations depicted in Figure 12-2a (just with the spread occurring in the two-dimensional mental space of our floor plane rather than the one-dimensional space of numerosity; Fyhn, Molden, Witter, Moser, & Moser, 2004; Hafting, Fyhn, Molden, Moser, & Moser, 2005; Moser, Kropff, & Moser, 2008). That is, approximate number representations obey the same principles of "noisy" approximate coding that operate quite broadly throughout the mind/brain.

This point is worth highlighting because it invites you to recognize that, whatever theory you end up preferring for approximate number system representations, that theory must make use of constructs that can apply quite broadly across cortical and subcortical representations. The differences in response times, error rates, and internal confidence that we noted during our discussions of Figures 12-1a and 12-1b have also been observed for the vast majority of the psychological dimensions that humans and other animals represent (e.g., scalar variability and ratio-dependent performance for time, number, distance, flavor concentration, electric shock, perceived weight, density, viscosity; Cantlon, Platt, & Brannon, 2009; Gescheider, 1997; Odic, Im, Eisinger, Ly, & Halberda, under review). Many psychological dimensions rely on coding schemes based on scalar variability and internal confidence (signal fidelity) that operate similarly to ANS representation.

In Figure 12-2a, as the number of items in an array presented to an observer increases from 4 to 10, the standard deviation of the bell-shaped curves that represent the numerosity increases, resulting in a flattening and spreading of the activations (note the peakier curve for approximately-4 and the broader curve for approximately-9 in Figure 12-2a). This increase in spread with increasing number is the basis for the hallmark properties of the ANS and, as discussed previously, is similar to discrimination in many other dimensions (e.g., brightness, loudness), discrimination dependent on ratio and not their absolute number (i.e., scalar variability, or Weber's law, described later). When you are trying to discriminate one numerosity from another using the Gaussian representations in Figure 12-2a, the more overlap there is between the two Gaussians being compared, the less accurately they can be discriminated.

Critically, the overlap between Gaussian distributions is also the source for the differences in accuracy, RT, and internal confidence we experienced when viewing Figures 12-1a and 12-1b. Numerosities that are closer together have more overlap in their curves on the mental number line, making them harder to separate from each other to determine which collection is more numerous. Ratios that are closer to 1, where the two numbers being compared are closer (e.g., Figure 12-1b), give rise to Gaussian ANS representations with greater overlap, resulting in poorer and slower discrimination (i.e., "ratio-dependent performance")—e.g., it feels easier to decide that there are more black dots than white dots when looking at Figure 12-1a than at Figure 12-1b (and observers would make fewer errors, and decide faster, when shown Figure 12-1a than Figure 12-1b). Looking at the curves, and their overlap, in Figure 12-2a helps you to picture why errors, response times, and internal confidence may change as the numerosities being compared become larger and closer in proportion.

To see how the bell-shaped representations of the ANS in Figure 12-2a can predict differences in errors, response times, and internal confidence, consider that the curve for approximately-5 in Figure 12-2a is broader than the curve representing approximately-4 (i.e., approximately-5 has a larger standard deviation than approximately-4). These two curves are fairly easy to visually tell apart in Figure 12-2a. But, as one increases in number (i.e., as one moves right in Figure 12-2a), the curves become more and more similar looking (e.g., is curve 9 higher and skinnier than curve 10, or do they look pretty much the same?). As the ANS representations become more similar—, i.e., as there is more overlap between the representations of the two numerosities to be discriminated—discrimination becomes harder, is more error-prone, and takes longer.[4] These bell-shaped representations predict that discrimination should smoothly become more and more difficult as the two numerosities become more and more similar.

In the ANS, it is not simply that larger numbers are harder to discriminate across the board. For example, an observer's performance at discriminating approximately-16 from approximately-20 (not shown in Figure 12-2) is predicted to be identical (in error rate, response time, and internal confidence) to the observer's performance at discriminating approximately-8 from approximately-10—as both of these trials would involve the same ratio (i.e., $10/8 = 1.125 = 20/16$). Although the curves for approximately-16 and approximately-8 do not have the same overall shape (e.g., the curve representing approximately-16 would be broader and flatter than the curve representing

4. This example based on the height and skinniness of the curves is simply to generate the intuition that discrimination becomes harder as the curves become more similar. Actual discrimination in the ANS is not based on the heights of the curves, but on the similarity of the activations elicited by the two sets of quantities (shown graphically in Figure 12-2a) and the amount of overlap between the two curves representing these numerosities.

approximately-8), it is the amount of overlap between the curves being compared that determines error rates, response times, and internal confidence. Because the standard deviation (SD) of the curves increases linearly with the mean ($SD = \mu * w$), the curves representing approximately-8 and approximately-10 will overlap in area to the same extent that approximately-16 overlaps with approximately-20.[5]

Behavioral performance in tasks that engage the ANS is richly textured and exquisitely well structured. What do we mean by this? Observers don't simply "do a bit worse" as the numbers become more similar; nor do they feel "just a bit less confident." Rather, each observer's error rate, response time, and internal confidence are exquisitely well predicted by the bell-shaped representations in Figure 12-2a, and the changes in observers' performance as a function of trial difficulty is very systematic. This systematicity is what any candidate theory of approximate number representations must account for.

There are, however, numerous misunderstandings about Weber's law (i.e., ratio-dependent performance) and especially the Weber fraction (w), which indexes individual differences in ANS accuracy and internal confidence. In what follows, we elaborate on the nature of the Weber fraction and go through some of the most common misconceptions about it, including that the Weber fraction indexes just-noticeable differences, that it is defined as 75% accuracy, etc.

How to Think of a Weber Fraction (w) in the Approximate Number System (ANS)

What is a Weber fraction (w), and what does it tell us about an observer's approximate number system (ANS) representations? Some common misunderstandings of a Weber fraction include that it is (1) the fraction by which a stimulus with numerosity n would need to be increased in order for a subject to detect and report the direction of this change resulting in 75% correct performance across trials (i.e., that it is the "difference threshold" or the "just noticeable difference," JND), (2) the smallest ratio at which subjects will be significantly above chance in a numerical discrimination task, and (3) the midpoint between subjective equality of two collections and asymptotic performance in numerical discrimination. Rather, the Weber fraction is all of these things, and it is also simpler, more abstract, and more basic than any of these. After illustrating some problems with the above views, we sketch a proposal that the Weber fraction can be understood as an internal scaling factor that

5. Note also that it is the numerical similarity between the sets that is important for determining how difficult a trial might be, and not their absolute size. Bigger is not always harder; it depends on the numerical distances involved. For example, 7 black versus 8 gray dots is a *harder* trial than 17 black versus 30 gray dots—because the ratio 30:17 is larger than the ratio 8:7. This is sometimes called the "size effect."

indexes the amount of internal precision (i.e., signal fidelity) of every approximate number representation, and that the Weber fraction, so understood, can be used to determine the standard deviation of every numerosity representation within the ANS, and can turn knowledge of any one approximate number representation into any other approximate number representation.

Consider Figure 12-2a to represent the ANS number representations for a particular individual who has a Weber fraction = 0.125. In the following sections, we describe what role this number (0.125) is taken to play by each of the four conceptualizations listed earlier. In the end, we suggest that understanding a Weber fraction to be an internal scaling factor indexing the internal precision, confidence, or signal fidelity of a person's approximate number thoughts is the most valuable and true conceptualization. We suggest that this number (0.125), so understood, tells us how imprecise, or "noisy," a person's approximate number thoughts are.

The Weber Fraction Is Not a Just Noticeable Difference (JND)

If you present the hypothetical subject (whose ANS representations are depicted in Figure 12-2a) with the tasks we did with Figures 12-1a and 12-1b (i.e., the task of determining which of two collections has the greater number of dots) on a trial where there are 16 gray dots, this subject would require an increase of 2 dots from this standard ($n_1 = 16$; $16 \bullet .125 = 2$; $n_2 = 16 + 2 = 18$) in order to respond that black ($n_2 = 18$) is more numerous than gray ($n_1 = 16$) on 75% of the trials that present these two numerosities.[6] That is, a subject's Weber fraction can be used to determine the amount by which you would need to change a particular stimulus in order for the subject to correctly determine which number was larger on 75% of the trials (where chance = 50%). Conceived in this way, the Weber fraction describes a relationship between any numerosity and the numerosity that will consistently be discriminated from this standard. This gives one way of understanding why you might choose 75% correct performance; however, to specify what "consistently discriminated from" might mean, you could also choose some other standard (e.g., 66.7% correct, or any other percent above 50%). From this point of view, which is often the dominant one taught in psychophysics, the point is to estimate the steepness of the linear portion of the psychometric function, or the slope of the linear rising portion of this function (depicted in Figure 12-2b), and 66.7% would work for such purposes just as well as 75% or 80%.

However, as we will see below, the seemingly arbitrary reasons for choosing 75% correct as an index of performance are somewhat justified once we understand the mathematical relationship that holds between correct discrimination performance, the Weber fraction (w), and the standard deviations of the underlying Gaussian representations.

6. Note, we use "•" throughout to indicate multiplication.

The Weber Fraction Is Not the Smallest Discriminable Ratio

Some readers, more familiar with research on the acuity of the ANS in infants (Izard et al., 2009; Libertus & Brannon, 2009; Lipton & Spelke, 2003; Xu & Spelke, 2000; Xu, Spelke, & Goddard, 2005) and less familiar with the literature on adult psychophysics, may have come to believe that a Weber fraction describes the ratio below which a subject will fail to discriminate two numerosities (e.g., 6-month-olds succeed with a 1:2 ratio and fail with a 2:3 ratio; Xu et al., 2005). This suggests a categorical interpretation of the Weber fraction (e.g., a threshold where you will succeed if a numerical difference is "above threshold" and fail if it is "below threshold"). That is, some may have come to believe that performance should be near perfect with ratios easier than a subject's Weber fraction and at chance for ratios harder than a subject's Weber fraction.

Categorical performance, however, is not observed in typical performance where a large number of trials test a subject's discrimination abilities across a wide range of ratios (Halberda & Feigenson, 2008; Halberda et al., 2012; Piazza et al., 2010). In such cases, behavioral performance shows a smooth improvement from a ratio of 1 (where $n_1 = n_2$ and there is no correct answer) toward increasing accuracy; and not a "step function" from at-chance performance below the Weber fraction to above-chance performance above the Weber fraction.

Consider again the simple task of being briefly shown a display that includes some black and white dots and being asked to determine on each flash if there were more black or more white dots. Percent correct on this numerical discrimination task is not a step function with poor performance "below threshold" and good performance "above threshold," but rather is a smoothly increasing function from near-chance performance to consistent success. This performance and the range of individual differences, gathered from more than 10,000 subjects between the ages of 8 and 85 years of age participating in this type of numerical discrimination task, can be seen in Figures 12-3a and 12-3b.

The actual behavioral data from subjects seen in Figure 12-3a, and the modeled ideal behavior seen in Figure 12-2b, suggest that the subjects will always be above chance no matter how small the difference between n_1 and n_2 (e.g., in theory, even a baby will be "above chance" at seeing that 10,001 black dots is numerically more than 10,000 gray dots; see Green & Swets, 1966); what changes is not whether an observer will succeed or fail to make a discrimination but rather the number of trials an experimenter would have to run in order to find a statistically significant difference in performance on the most difficult trials. Consider that the region nearest to equality (a ratio of 1) is the region of most rapid improvement in every observer's performance (e.g., Figures 12-3a and 12-2b). That is, subjects' performance shows more improvement when the ratios being tested increase from 1.01 to 1.1 than they show when the ratios increase from 1.33 to 1.4. There are two take-home points that we'd like to stress: (1) even a baby should be able to tell that 21 black dots is numerically more than 20 gray dots (what changes is the number of trials we'd have to run to be able to show that

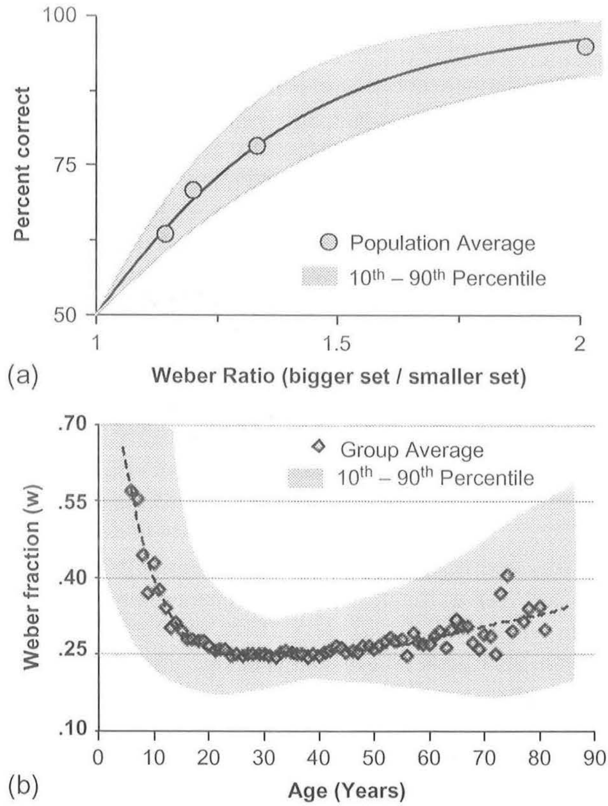


FIGURE 12-3 ANS performance from more than 10,000 participants. (a) ANS discrimination smoothly increases with ratio; (b) Weber fractions initially become better (i.e., become lower) with age, and eventually plateau around age 30, and subsequently slowly become worse. Note that these graphs present previously unpublished data collected on testmybrain.org and panamath.org during 2008 (see Halberda et al., 2012).

infants detect this difference), and (2) it is at the hardest ratios (e.g., 1.01 versus 1.1) that we see the most rapid improvements in numerical discrimination performance (and not at some "threshold" or fraction that changes from "at chance" to "above chance").[7]

7. However, we would also like to note that, within the practical limits of testing real babies, the infant literature's method of looking for a change from at-chance performance to above-chance performance is a quite reasonable approach. It allows one to *roughly* locate the Weber fraction of subjects who, like infants, cannot participate in the large number of trials it takes to achieve the smooth data seen in Figure 12-3a. We have published papers that use this kind of approach, and it is a fine thing to do. But we're suggesting that it would be best if we do not allow such practical concerns to inspire a faulty foundation on which to grow our theory-based intuitions of what is possible and impossible for the ANS and other magnitude representations.

Even for infants, then, and untrained observers, performance in numerical discrimination tasks should not shoot up from "at chance" for harder ratios to "significantly above chance" at easier ratios. Melissa Libertus and colleagues have ingeniously demonstrated that infants looking time to numerically varying stimuli can reveal a smoothly graded function of increasing looking that is quite similar to what is seen in Figure 12-3a (Libertus & Brannon, 2009, 2010). In these tasks, infants are shown a display of dots at a particular ratio, and their looking time to the display is measured; the amount of time infants spend looking at the display is a continuous function dependent on the ratio: as ratios get harder and harder, infants gradually look less and less. This work highlights, in dramatic fashion, that misunderstanding a Weber fraction to indicate something about a change from chance performance (or an "inability to distinguish") to a sudden ability to distinguish numerically varying stimuli will generate the wrong intuitions (i.e., in theorists, teachers, and students).

The Weber Fraction Is Not the Midpoint between Subjective Equality and Asymptotic Performance

One common approach to localize the Weber fraction at some point along the smoothly increasing curve in Figure 12-3a is at 75% accuracy—the midpoint between subjective equality of the two numerosities being compared (without biases, occurring at a ratio = 1, where $n_1 = n_2$) and asymptotic performance (typically occurring nearing 100% correct, although asymptotic performance could be lower in unskilled subjects, resulting in a midpoint that falls at a percent correct lower than 75%; for example, see Halberda & Feigenson, 2008).[8] Hence, to calculate a Weber fraction, a researcher may take the ratio at which the observer performed at 75% and then subtract 1 (e.g., if 75% performance is at a ratio of 1.25, then the w is estimated at 0.25).

If observers behave optimally and if the Weber fraction is within a particular range, this shorthand does produce the correct value. In Figure 12-2b, we have drawn the expected percent correct for the ideal subject in Figure 12-2a whose Weber fraction (w) = 0.125 as derived by a model from classical psychophysics. This idealized subject would perform at 75% around ratio 1.128.

There are two challenges, however, with conceptualizing the Weber fraction in this manner. First, the mathematical relation between w (conceptualized as a scaling factor) and the ratio at 75% is not constant: whereas an observer with a w of 0.125 will perform at about 75% around ratio 1.125, an observer with a w of 0.5 will perform at around 75% around a ratio of 2.0, not 1.5!

8. Typically, behavioral performance will cross 50% at ratio = 1 for an observer who has no bias to choose black or white and who is simply guessing at chance = 50% when $n_1 = n_2$ (#black = #gray); and it may never reach 100% no matter how easy the trials become (e.g., because everyone has some tendency to make a miss-hit on the response keys from time to time, even if merely from sheer boredom with all those dots).

Second, understanding of the Weber fraction as the midpoint between subjective equality and asymptotic performance misses the deeper continuous nature of discrimination within the ANS. For example, this focus on 75% has led many researchers throughout history to believe that accuracy is the theoretical variable they are hoping to measure. For example, many readers may be familiar with "staircase" methods or adaptive procedures that adjust trial difficulty in response to the subject's performance in an attempt to focus the majority of trials on a position where the subject is at 75% correct (as if this point were of special importance). As we see in the next section, there is nothing at all special about 75% correct performance. The Weber fraction (w), properly understood as a scaling factor for determining internal variability for every approximate number representation, perfectly predicts performance at 75% correct, or 85%, or 51.3756% correct, and everything in between; and it determines performance along the entire smoothly improving curve seen in Figures 12-2b and 12-3b.[9]

Weber Fraction Conceptualized as a Scaling Factor

Let us consider a fourth way of understanding the Weber fraction: as a scaling factor that indexes the amount of "noisiness" surrounding every numerical representation of the ANS.

Consider again the Gaussian curves in Figure 12-2a. The spread of each successive numerosity from 4 to 10 is steadily wider than the numerosity before it. This means that the discriminability of any two numerosities is a smoothly varying function, dependent on the ratio between the two numerosities to be discriminated. In theory, such discrimination is never perfect because any two numerosities—no matter how distant from one another—will always share some overlap. At the same time, discrimination will never be entirely impossible, so long as the two numerosities are not identical, because any two numerosities, no matter how close (e.g., 67 and 68), will always have some nonoverlapping area where the larger numerosity is detectably larger (Green & Swets, 1966). Correct discrimination may occur on only a small percentage of trials if the two sets are very close in number, but it will never be impossible (up to the limits of the sensory detector). This motivates the intuition that percent correct in a numerical discrimination task should be a smoothly increasing function from the point of subjective equality to asymptotic performance. The smooth increase in percent correct as a function of

9. For those interested in practical concerns, the most reliable and stable performance for human subjects, where trials are neither too easy nor too hard, occurs at around 86% correct performance (let's call it the "Goldilocks position") and not at 75% correct (this factoid garnered from modeling work in our lab, and our practical experiences testing subjects across a wide range of ability levels and ages, and informed by conversations with the great and stimulating Zhong-Lin Lu). So, even for practical reasons (beyond theoretical concerns) we should not focus on 75% as something special.

ratio is no accident. It is the smoothly increasing spread in the underlying Gaussian representations depicted in Figure 12-2a that is the source of the smoothly increasing "Percent Correct" ideal performance in Figure 12-2b.

Noting the smoothly increasing spread of the Gaussian representations in Figure 12-2a might motivate you to ask what is the parameter that determines the rate of increase in standard deviation with numerosity, and what determines the amount of spread in each Gaussian representation on the mental number line? In fact, it is the Weber fraction that determines the spread of every single representation on the mental number line by the following formula ($SD_{n1} = \bar{x}_{n1} \bullet w$). The standard deviation (SD) of the Gaussian bell-shaped curve representing any particular numerosity on the mental number line is the central tendency for that representation ($\bar{x}_{n1}$) multiplied by the Weber fraction (w).

Why is this the case? Well, intuitively, it is the standard deviations of the underlying Gaussian representations that determine the amount of overlap between the curves that represent any two numerosities, and it is the amount of overlap between the numerosities that determines how well any two numerosities can be discriminated. The categorical views of the Weber fraction as a kind of threshold between successful discrimination and failure, or as the midpoint between subjective equality and asymptotic performance, choose to focus on only one particular point of what is actually a continuous and smooth function of increasing success at discrimination. As a result, this entire function is determined by the Weber fraction because this parameter describes the standard deviations of every single numerosity representation in the ANS—and therein the degree of overlap between any two numerosities on the mental number line.

The Weber fraction (w) is the constant that describes the amount of precision for each observer's ANS number representation. It is a scaling factor by which you could take any one of the curves in Figure 12-2a and turn it into any of the other curves in Figure 12-2a in an accordion-like fashion. In the linear model depicted in Figure 12-2a, the analog representation for any numerosity (e.g., $n=7$) is a Gaussian random variable with a mean at n (e.g., $n=7$) and a standard deviation of ($n \bullet w$).[10,11] This means that for a subject who has a Weber fraction of 0.125, the ANS representation for $n=7$ will be a bell-shaped normal curve with a mean of 7 on the mental number line and a standard deviation of $0.875=0.125 \bullet 7$. By substituting any number you like for n, you can easily determine the shape of the underlying ANS representation without ever having the observer engage in a numerical

10. Note that signal compression or expansion is also important because it can change the position of representations along the mental number line (for detailed discussion, see Odic et al., under review)—a detail that does not concern us for the present moment.

11. Note also that the relationship of the Weber fraction (w) to internal confidence is also true for a logarithmic model of numerosity representation, with any differences in details not relevant for the present discussion.

discrimination task that compares two numbers. This illustrates the power of understanding the Weber fraction as an index of signal fidelity, internal confidence, or internal noise. Rather than simply telling us something about how well a subject will do at discriminating two numbers "near their just noticeable difference," the Weber fraction (w) tells us the shape and overlap of every single number representation along a mental number line. The Weber fraction is about all of the representations, not just the ones "near threshold."

Understood in this way, the Weber fraction is not even specific to the task of numerical discrimination; indeed, it is wholly independent and prior to discrimination. An animal that, bizarrely, could only represent a single numerical value in its ANS (e.g., could represent only *approximately-7* and no other numbers) and could therefore never discriminate 12 from any other number (i.e., could not even perform a numerical discrimination task) would nonetheless have a Weber fraction, and we could measure it!

Meet Justin The Rat

In this section, we want to briefly discuss the beautifully limited mind of an animal named "Justin The Rat," which, strangely, can only represent the number *approximately-7* and no other number. Justin The Rat can represent all the other things that we represent (e.g., dots, colors), but for numbers, he has only one thought, and that is the thought *approximately-7*.

> Question: Hey, Justin The Rat, how many food pellets did you just eat?
> Answer: *Approximately-7*.
> Question: On another topic, Justin The Rat, surely you do not believe in God?
> Answer: Well, not in an interventionist Christian god, if that's what you mean.
> Question: Dear Justin The Rat, on a scale from 1 to 10, with 10 being "smoking hot" and 1 being "let's not talk about this," how sexy am I really?
> Answer: *Approximately-7*.

You get the idea.

How well would Justin The Rat do if we asked him to choose which array has more dots while showing him 16 gray dots and 18 black dots? It may seem predestined that Justin The Rat would be terrible at a numerical discrimination task involving two sets of dots, each with more than 7 dots—owing to his unique brain abnormality that limits his numerical thoughts to *approximately-7* and nothing else. For instance, how would we teach him such a task or measure his performance? And, is it even possible to have a living creature with numerical cognitive abilities so impaired? Does Justin The Rat *have* a Weber fraction (w), and how would we measure it? To answer these questions, we invite you to take our earlier, technical, sections as a point of departure.

As we have seen, a Weber fraction can be understood to be a scaling factor that determines the standard deviation of the bell-shaped curves representing each and every number representation in a subject's approximate number system (ANS). Numerical discrimination tasks (e.g., Figures 12-1a and 12-1b) are not the only way of measuring this type of internal scaling parameter.

Although production tasks (such as the "tap your finger *n* times" task; Whalen, Gallistel, & Gelman, 1999) and discrimination tasks (such as the "who has more" task; e.g., Figure 12-1) have often been discussed separately, they measure theoretically identical aspects of ANS representations. For a "tap your finger *n* times" task, researchers generate a measure of the coefficient of variation (CoV or CV), which is the standard deviation of the number of presses divided by the mean number of presses. For example, ask a subject to press a button 9 times too quickly for explicit counting while saying the word "the" to further block verbal counting. The result will be a bell-shaped distribution of responses; the subject will most often press 9 times, but will also sometimes press 8 or 7, and sometimes press 10 or 11 or 12, etc. Graphing the number of instances where the subject presses 7, 8, 9, 10, 11, etc., times when requested to press 9 times will reveal a smooth bell-shaped curve centered around 9 (similar to the curve for 9 in Figure 12-2a). Take the standard deviation of this bell-shaped curve and divide it by the mean of this curve to return the CV for this subject (i.e., $CV_{n9} = SD_{n9}/\overline{x}_{n9}$). For this task, which also engages the ANS, CV is expected to be constant across all numbers probed. That is, ask the same subject to do the study again pressing, e.g., $n = 14$ times, build a similar-looking (but fatter) bell-shaped curve centered around 14, divide the standard deviation of this curve by the mean of this curve, and you should get the same CV that you got for the version of the task in which the requested number of presses was 9 (Cordes et al., 2001; Whalen et al., 1999); $CV_{n14} = SD_{n14}/\overline{x}_{n14}$; and, $CV_{n14} \approx CV_{n9}$.

The source of the bell-shaped curves in a numerical production task is not, in theory, simply mispresses or mistakes (a curve built out of mispresses would not be a bell-shaped Gaussian, but a more narrow, binomial, non-Gaussian, curve; Cordes et al., 2001). The source of the bell-shaped curves, and the fatness of the curves, is the variability in the underlying representations of the ANS. And this leads to a little-remarked-upon identity: an observer's CV and his or her Weber fraction (w) should, theoretically, be identical numbers.

Note, $CV_n = SD_n/\overline{x}_n$. And, as we mentioned previously, the SD of the underlying ANS representation for any number can be determined using a subject's Weber fraction (w) as an internal scaling parameter, i.e., $SD_n = \overline{x}_n \bullet w$. Rearrange this equation and you get $SD_n/\overline{x}_n = w$, the same equation that we use to calculate CV (i.e., $SD_n/\overline{x}_n = CV_n$). That is, $CV = w$.

This identity makes intuitive sense upon reflection. As subjects try to tap their finger quickly in a numerical production task, they give up on verbal counting and allow their ANS to assess when the target number of taps has been reached. But these ANS representations are "noisy," leading observers to sometimes tap too many times and sometimes tap too few times over the course of many trials. In this way, the source of the errors in a "tap your finger" task is the noisy representations of the ANS. The source of the errors in a numerical discrimination task (e.g., Figures 12-1a and 12-1b) is also the noisy representations of the ANS. The coefficient of variance (CV) and Weber fraction (w) are two ways of estimating the imprecision, or variability, in these underlying representations. And so, CV and w are two ways of measuring the same thing.

Thus, to estimate the Weber fraction for Justin The Rat—an animal that, strangely, can represent only *approximately-7*—train him to press a button 7 times, run many trials, calculate CV, and CV = w (for an alternative method, see Odic, Im, et al., under review, and visit www.panamath.org/psimle). Here, you have found the Weber fraction for an animal without ever having that animal compare two numbers or see two collections. This is an illustration of the inductive power of understanding the Weber fraction (w) to be an internal scaling factor. A Weber fraction need not require an understanding of failure or success at numerical discrimination, nor even the ability to make a numerical discrimination. Rather a Weber fraction (w) is simply a way of indexing the internal precision (aka signal fidelity, internal confidence, noise) in a person's ANS approximate number representations.[12]

How a Weber Fraction (*w*) Indexes Individual Differences in ANS Precision

The inductive power of understanding the Weber fraction (w) to be an internal scaling factor is further highlighted when we compare the Weber fractions of different individuals. Individuals differ in the precision of their ANS representations. Some people have less precise approximate number representations, and some people have more precise representations (Halberda & Feigenson, 2008; Halberda et al., 2012). In Figure 12-4a, we have illustrated some idealized curves that display the underlying ANS representations for a subject whose $w = 0.125$ and, in Figure 12-4b, for a subject whose $w = 0.20$. Crucially, you can see that the subject in Figure 12-4b has a greater degree of overlap between the bell-shaped curves of their ANS representations than the subject in Figure 12-4a (recall, a bigger Weber fraction means more noise and fatter curves). It is this overlap that leads to differences in internal

12. In full disclosure, production tasks and discrimination tasks may not always be measuring the same thing, because in fact, it is unlikely that any psychological task is measuring only one thing. No matter how simple you make the task, it is likely that many different psychological factors are required for encoding, response generation, and decision making. As such, in practice, measured CV will not perfectly predict measured w. A scientifically productive question CV might be, "How might the differences in measured estimates of CV and w help us determine the variety of psychological variables these tasks have in common and those that they have distinctly?"

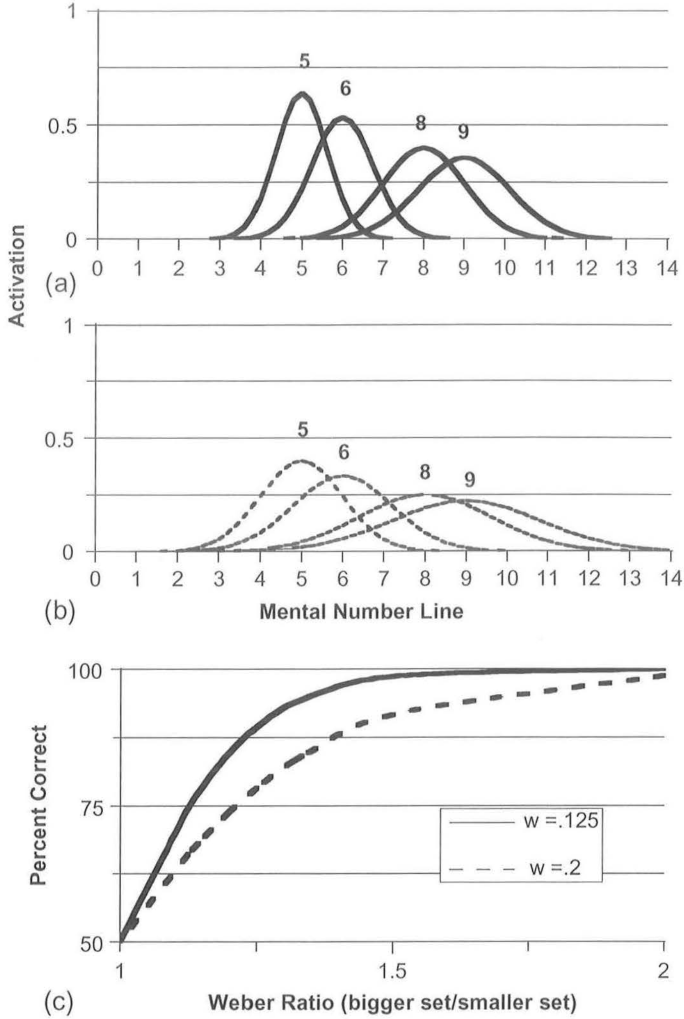


FIGURE 12-4 Individual differences in the ANS. The difference between an individual with a better (lower) Weber fraction in (a) and an individual with a worse (higher) Weber fraction in (b) is entirely in the variability of the ANS representations: higher overlap between the representations results in lower accuracy (illustrated in [c]) and lower confidence and higher RT.

confidence, error rates, and response times—and to the difficulty in discriminating two stimuli that are close in numerosity. The hypothetical subject in Figure 12-4b would have poorer performance than the subject in Figure 12-4a in a numerical discrimination task (e.g., Figures 12-1a and 12-1b).

The ideal performance in Figure 12-4c also shows the smooth gradual increase in percent correct as a function of ratio that we have been discussing. In Figures 12-3a and 12-3b, we saw data from more than 10,000 individuals who played a numerical discrimination task online. Every one of the more than 10,000 observers in this sample obeyed this kind of gradual increase in percent correct (seen in Figure 12-3a) from a ratio of 1 (where the number of black and gray dots are equal) to easier ratios like 2 (where there might be 20 black dots and 10 gray dots; 20/10 = 2). What changes from observer to observer is how steep the left side of the performance curve is (NB, you can see this difference in Figure 12-4c for the two hypothetical subjects who differ in their Weber fraction).

In Figures 12-3a and 12-3b, individual differences are shown by indicating the range of performance from the lower 10^{th} to the upper 90^{th} percentile rank of the more than 10,000 observers (i.e., the lower bound of the gray-shaded region in Figure 12-3b indicates the average performance of the 90^{th} percentile group [i.e., best], and the upper bound of the gray-shaded region indicates the average performance of the 10th percentile group [i.e., worse]; note that the upper and lower bounds are reversed for Figure 12-3a because higher percent correct translates to lower Weber fraction (w), i.e., lower internal noise). Figure 12-3b shows how the average Weber fraction improves over development.

A steeper, quicker rise in the psychometric function (Figures 12-4c and 12-3a) indicates better sensitivity, better discrimination abilities, more precise ANS representations (e.g., sharper bell-shaped humps in the ANS, with less "noise"; smaller standard deviations for each hump), and this is indexed by the subject having a smaller Weber fraction (Figures 12-4a, 12-4b, and 12-3b) (i.e., a smaller Weber fraction indicates less noise in the underlying ANS representations).

The values for the Weber fractions in Figure 12-4 have been chosen so as to illustrate another value of understanding the Weber fraction to be an internal scaling factor: it empowers comparisons across individuals and formal models of individual differences. Converting the Weber fraction for each of these subjects into the nearest whole number fraction reveals that the Weber fraction for the subject in Figure 12-4a is 8:9 and for the subject in Figure 12-4b is 5:6 (i.e., 9/8 = 1.125; 6/5 = 1.20). Investigating the Gaussian curves in Figure 12-4a and 12-4b reveals that the bell-shaped curves for the numerosities 8 and 9 for the subject in Figure 12-4a are identical in shape to the bell-shaped curves for the numerosities 5 and 6 for the subject in Figure 12-4b. This too is no accident. The only parameter that has been changed in the construction of Figures 12-4a and 12-4b is the Weber fraction for the subject. This single parameter determines the spread in the curves that represent every possible numerosity in the ANS of each subject.

In this way, understanding the Weber fraction to be an internal scaling factor that determines the spread of every ANS number representation not only empowers us to compare one number representation to another within a particular subject (e.g., the lesson we learned from Justin The Rat), but also empowers us to compare across individuals and to create mathematically tractable predictions about how the ANS representations of one person (e.g., the subject in Figure 12-4a) relate to the ANS representations of another (e.g., the subject in Figure 12-4b). This is not the case for any other estimate of individual differences that you might prefer to use (e.g., percent correct, average response time, the slope of error rate as a function of ratio, the slope of response time as a function of ratio), although these may be used as rough approximations (e.g., just like subtracting 1 from the 75% ratio can, in some circumstances, be used as an approximation).

Two important goals of psychology are to measure and to understand the sources of individual differences in a wide range of social behaviors and cognitive abilities. A valuable approach to approaching these challenges is to develop a formal model of the particular aspect of the psychological system that you hypothesize is different from one person to another—such as the precision and accuracy of the bell-shaped representations of the ANS. When we understand a Weber fraction (w) to be an internal scaling factor that indexes the precision of each person's approximate number thoughts, we find that this allows us to directly translate (in a formal sense, seen graphically in Figures 12-4a to 12-4c) from one individual's ANS precision to another individual's ANS precision; and to build specific proposals for the shape and activation of each numerical representation within each individual's ANS.

There remains important work to be done, both practical and theoretical, to ensure that we are correctly measuring subjects' Weber fractions (e.g., how much display time is optimal? Does the Weber fraction change if we present auditory stimuli rather than visual stimuli?). Also, we must strive to ensure that our formal models of a Weber fraction reflect the actual behavior and neuronal activity of our subjects of interest. This is an ongoing process for our research field, and we do believe there are major discoveries still to be made. But, we also believe that understanding a Weber fraction as a scaling factor is an important foundation for studying individual differences, and for beginning a journey of making new discoveries that will help us build more appropriate and accurate models of cognition.

THE RELATION BETWEEN THE WEBER FRACTION AND INTERNAL CONFIDENCE

One further lesson we can draw from our experiences with Figures 12-1a and 12-1b is the power of internal confidence to inform how we search and interact with the world. If you happen to still have children nearby, you might try asking which picture (Figure 12-1a or 12-1b) they think looks like the easier picture to answer without actually making the judgment. Which picture will it be easier to figure out "who has more?" We imagine that you would find that even without counting, children will judge that Figure 12-1a will be the easier trial. We believe that this ability to respond which figure would be easier to answer emerges from our sense of internal confidence for ANS questions and displays, which directly stems from the internal variability in the

Gaussian (bell-shaped) representations (whose value is scaled by the Weber fraction). In this way, the psychophysical model outlined above of the Weber fraction as a scaling parameter provides a unified explanation for the ratio signatures, individual differences, and the source of internal confidence in our approximate number decisions.

Importantly, notice that you and the child can answer this "easiness" question (and the "which are you more likely to make an error on" question asked earlier) *even before you figure out the correct answer*. That is, even without ever being told which color has more dots (i.e., before telling the child that "black has more" for both Figure 12-1a and Figure 12-1b), we seem to be able to tell that Figure 12-1a will be the easier image to answer which has more. Because our feeling of internal confidence occurs prior to our decision, we can use it in several ways. For example, it is a signal to "slow down" and be more careful about answering the question for Figure 12-1b. We might also have a sense that we should "look more closely at those dots on the lower left corner of Figure 12-1b before we answer to see if there are more black or gray dots down in that visually crowded region of the display." This means that our sense of internal confidence (and trial difficulty), generated by our ANS representations, can help us decide how to *approach* answering a question (e.g., where to allocate our attention, or when to be careful and take a second glance). Thus, the ANS is not simply in the business of giving us an answer to a numerical question; it is, perhaps more importantly, involved in helping us direct our limited attention and memory resources to help us make more effective decisions (Odic et al., 2012).

In this way, far from amounting to counterproposals to the importance of the ANS, some recent results revealing that observers are affected by stimulus factors such as size-conflicting stimuli (Dakin et al., 2011; Gebuis & Reynvoet, 2012; Szucs, Nobes, Devine, Gabriel, & Gebuis, 2013), spatially intermixed stimuli (Gebuis & Reynvoet, 2012; Price et al., 2012), or briefly flashed stimuli (Inglis & Gilmore, 2014) are all beautiful demonstrations of the importance of internal confidence, generated by ANS representations for guiding our numerical decisions. We hypothesize that observers rely on internal confidence from the ANS (which is sensitive to the context of stimulus presentation) to marshal their cognitive control abilities in order to respond more effectively to numerical stimuli, which vary wildly in their mode of presentation across contexts (e.g., sounds heard, objects seen, all at once or serially presented). The effects of stimulus presentation and, e.g., size/duration-controlled or size/duration-confounded stimuli are beautiful demonstrations of the importance of approximate number representations and internal confidence for numerical cognition.

Finally, internal confidence, in being fundamentally related to Weber fractions, is also an important individual difference that may be related to other cognitive abilities. An observer who has less precise ANS representations for number will feel somewhat confident that "black has more" for

Figure 12-1a, and may feel very low confidence that "black has more" for Figure 12-1b (and all observers are likely to take longer and to make more errors for Figure 12-1b than 12-1a). We theorize that this difference in internal confidence has a major impact on how we feel about mathematics across our entire lives and may be *the dominant source* for what we experience as, "I am (am not) a math person." Ongoing work in our lab is testing the relation between internal confidence and school math performance. We believe that an understanding of internal confidence can be a major unifying force for the study of approximate number representations. All of the work being done in this exciting field is valuable and relevant; e.g., every empirical paper reports findings that can help to refine our theories of approximate number representation. Rather than defending older ideas about how internal representations of approximate number might affect our mathematical thinking, we are excited to look to the future for new constructs that can unify across our older distinctions.

CONCLUSION

In this chapter, we have tried to promote understanding a Weber fraction (w) as a scaling factor that enables any ANS number representation to be turned into any other; or, equivalently, as an index of the amount of internal confidence a person experiences in his or her approximate number thoughts. Understood in this way, a Weber fraction does not require the commonsense notion of a "threshold" (i.e., a change from failure to success), and it does not generate the same kinds of confusions that this commonsense notion gives rise to. Additionally, this psychophysical model integrates with a variety of signatures that have been experimentally observed.

We believe that thinking about a Weber fraction as JNDs, critical ratios, of 75% performance has given rise to some confusions and that it is currently limiting our theorizing (e.g., the confusion that performance should change from chance performance at difficult ratios to above-chance performance at easy ratios, while, as shown in Figure 12-3, the actual performance of subjects does not look this way at all, but instead is a smoothly increasing function). It also does not promote the kind of understanding of the approximate number system (ANS) that highlights the systematic nature of variability throughout the system. Understanding a Weber fraction (w) to be a scaling factor—i.e., an estimate of signal fidelity across all possible ANS representations—promotes our understanding that variability inherent in ANS representations is not merely a bug but is rather a feature of our approximate number system.

The heart of the ANS (and the psychological experiences of number that it generates) is its ordinal and approximate character. Because the ANS displays scalar variability in its coding of numerosity, understanding the variability of any one ANS representation (e.g., through measuring CV) can be easily translated into an understanding of the internal variability, and internal confidence, for every single ANS number representation. Furthermore, understanding a Weber fraction (w) as a scaling factor also promotes our understanding of the systematic relationships that exist across individuals (e.g., the comparison of the two subjects in Figure 12-4).

When we understand the Weber fraction (w) to be an internal scaling parameter that indexes the amount of precision and internal confidence in each person's approximate number thoughts, we can begin to see many new doors for research begin to open—e.g., connections to math anxiety, stereotype threat, the feeling of "I'm just not a number person." Connections to executive functioning and the possibilities for interventions to improve number sense also come into a sharper focus. All these avenues have yet to be fully explored, and we are excited to be able to play just a small role in testing out some of these ideas.

REFERENCES

Barth, H., Starr, A., & Sullivan, J. (2009). Children's mappings of large number words to numerosities. *Cognitive Development, 24*, 248–264.

Booth, J. L., & Siegler, R. S. (2006). Developmental and individual differences in pure numerical estimation. *Developmental Psychology, 42*(1), 189.

Burr, D. C., & Ross, J. (2008). A visual sense of number. *Current Biology, 18*(6), 425–428.

Cantlon, J. F., & Brannon, E. M. (2006). Shared system for ordering small and large numbers in monkeys and humans. *Psychological Science, 17*(5), 401–406. http://dx.doi.org/10.1111/j.1467-9280.2006.01719.x.

Cantlon, J. F., Platt, M., & Brannon, E. M. (2009). Beyond the number domain. *Trends in Cognitive Sciences, 13*(2), 83–91. http://dx.doi.org/10.1016/j.tics.2008.11.007.

Castelli, F., Glaser, D., & Butterworth, B. (2006). Discrete and analogue quantity processing in the parietal lobe: A functional MRI study. *Proceedings of the National Academy of Sciences, 103*(12), 4693–4698. http://dx.doi.org/10.1073/pnas.0600444103.

Cordes, S., Gallistel, C. R., Gelman, R., & Latham, P. (2007). Nonverbal arithmetic in humans: Light from noise. *Attention, Perception, & Psychophysics, 69*(7), 1185–1203. http://dx.doi.org/10.3758/BF03193955.

Cordes, S., Gelman, R., Gallistel, C. R., & Whalen, J. (2001). Variability signatures distinguish verbal from nonverbal counting for both large and small numbers. *Psychonomic Bulletin & Review, 8*(4), 698–707.

Dakin, S. C., Tibber, M. S., Greenwood, J. A., Kingdom, F. A. A., & Morgan, M. J. (2011). A common visual metric for approximate number and density. *Proceedings of the National Academy of Sciences, 108*(49), 19552–19557. http://dx.doi.org/10.1073/pnas.1113195108.

De Smedt, B., Noël, M.-P., Gilmore, C., & Ansari, D. (2013). How do symbolic and non-symbolic numerical magnitude processing skills relate to individual differences in children's mathematical skills? A review of evidence from brain and behavior. *Trends in Neuroscience and Education, 2*(2), 48–55.

Dehaene, S. (1997). *The number sense.* New York: Oxford University Press.

Dehaene, S. (2003). The neural basis of the Weber–Fechner law: A logarithmic mental number line. *Trends in Cognitive Sciences, 7*(4), 145–147.

Dehaene, S., & Cohen, L. (1997). Cerebral pathways for calculation: Double dissociation between rote verbal and quantitative knowledge of arithmetic. *Cortex, 33*, 219–250.

Dehaene, S., Izard, V., Spelke, E., & Pica, P. (2008). Log or linear? Distinct intuitions of the number scale in Western and Amazonian indigene cultures. *Science*, *320*(5880), 1217–1220. http://dx.doi.org/10.1126/science.1156540.

Durgin, F. H. (1995). Texture density adaptation and the perceived numerosity and distribution of texture. *Journal of Experimental Psychology. Human Perception and Performance*, *21*(1), 149.

Feigenson, L., Libertus, M. E., & Halberda, J. (2013). Links between the intuitive sense of number and formal mathematics ability. *Child Development Perspectives*, *7*(2), 74–79.

Franconeri, S. L., Bemis, D. K., & Alvarez, G. A. (2009). Number estimation relies on a set of segmented objects. *Cognition*, *113*(1), 1–13.

Frank, M. C., Everett, D. L., Fedorenko, E., & Gibson, E. (2008). Number as a cognitive technology: Evidence from Pirahã language and cognition. *Cognition*, *108*(3), 819–824.

Fyhn, M., Molden, S., Witter, M. P., Moser, E. I., & Moser, M.-B. (2004). Spatial representation in the entorhinal cortex. *Science*, *305*(5688), 1258–1264.

Gallistel, C. R., & Gelman, R. (2000). Non-verbal numerical cognition: From reals to integers. *Trends in Cognitive Sciences*, *4*(2), 59–65.

Gebuis, T., & Reynvoet, B. (2012). The interplay between nonsymbolic number and its continuous visual properties. *Journal of Experimental Psychology. General*, *141*(4), 642.

Gescheider, G. A. (1997). *Psychophysics: The fundamentals* (3rd ed.). Mahwah: Lawrence Erlbaum Associates.

Gevers, W., Verguts, T., Reynvoet, B., Caessens, B., & Fias, W. (2006). Numbers and space: A computational model of the SNARC effect. *Journal of Experimental Psychology. Human Perception and Performance*, *32*(1), 32–44. http://dx.doi.org/10.1037/0096-1523.32.1.32.

Gilmore, C., Attridge, N., Clayton, S., Cragg, L., Johnson, S., Marlow, N., et al. (2013). Individual differences in inhibitory control, not non-verbal number acuity, correlate with mathematics achievement. *PLoS ONE*, *8*(6), e67374.

Gordon, P. (2004). Numerical cognition without words: Evidence from Amazonia. *Science*, *306*(5695), 496.

Green, D. M., & Swets, J. A. (1966). *Signal detection theory and psychophysics*. Los Altos Hills, CA, USA: Peninsula.

Hafting, T., Fyhn, M., Molden, S., Moser, M.-B., & Moser, E. I. (2005). Microstructure of a spatial map in the entorhinal cortex. *Nature*, *436*(7052), 801–806.

Halberda, J., & Feigenson, L. (2008). Developmental change in the acuity of the "number sense": The approximate number system in 3-, 4-, 5-, and 6-year-olds and adults. *Developmental Psychology*, *44*(5), 1457–1465. http://dx.doi.org/10.1037/a0012682.

Halberda, J., Ly, R., Wilmer, J. B., Naiman, D. Q., & Germine, L. (2012). Number sense across the lifespan as revealed by a massive Internet-based sample. *Proceedings of the National Academy of Sciences*, *109*(28), 11116–11120. http://dx.doi.org/10.1073/pnas.1200196109.

Halberda, J., Sires, S. F., & Feigenson, L. (2006). Multiple spatially overlapping sets can be enumerated in parallel. *Psychological Science*, *17*(7), 572–576. http://dx.doi.org/10.1111/j.1467-9280.2006.01746.x.

Hauser, M. D., Tsao, F., Garcia, P., & Spelke, E. S. (2003). Evolutionary foundations of number: Spontaneous representation of numerical magnitudes by cotton-top tamarins. *Proceedings of the Royal Society of London, Series B: Biological Sciences*, *270*(1523), 1441–1446. http://dx.doi.org/10.1098/rspb.2003.2414.

Hurewitz, F., Gelman, R., & Schnitzer, B. (2006). Sometimes area counts more than number. *Proceedings of the National Academy of Sciences*, *103*(51), 19599.

Im, H. Y., Zhong, S., & Halberda, J. (2013). Biases in human number estimation are well-described by clustering algorithms from computer vision. *Journal of Vision, 13*(9), 829.

Inglis, M., & Gilmore, C. (2014). Indexing the approximate number system. *Acta Psychologica, 145*, 147–155.

Izard, V., & Dehaene, S. (2008). Calibrating the mental number line. *Cognition, 106*(3), 1221–1247. http://dx.doi.org/10.1016/j.cognition.2007.06.004.

Izard, V., Sann, C., Spelke, E. S., & Streri, A. (2009). Newborn infants perceive abstract numbers. *Proceedings of the National Academy of Sciences, 106*(25), 10382–10385. http://dx.doi.org/10.1073/pnas.0812142106.

Jordan, K. E., & Brannon, E. M. (2006). The multisensory representation of number in infancy. *Proceedings of the National Academy of Sciences of the United States of America, 103*(9), 3486.

Laming, D. (1997). *The measurement of sensation*. Oxford University Press. Retrieved from, http://www.oxfordscholarship.com/view/10.1093/acprof:oso/9780198523420.001.0001/acprof-9780198523420.

Le Corre, M., & Carey, S. (2007). One, two, three, four, nothing more: An investigation of the conceptual sources of the verbal counting principles. *Cognition, 105*(2), 395–438.

Libertus, M. E., & Brannon, E. M. (2009). Behavioral and neural basis of number sense in infancy. *Current Directions in Psychological Science, 18*(6), 346.

Libertus, M. E., & Brannon, E. M. (2010). Stable individual differences in number discrimination in infancy. *Developmental Science, 13*(6), 900–906. http://dx.doi.org/10.1111/j.1467-7687.2009.00948.x.

Lipton, J. S., & Spelke, E. S. (2003). Origins of number sense large-number discrimination in human infants. *Psychological Science, 14*(5), 396–401. http://dx.doi.org/10.1111/1467-9280.01453.

Mazzocco, M. M. M., Feigenson, L., & Halberda, J. (2011). Impaired acuity of the approximate number system underlies mathematical learning disability (Dyscalculia). *Child Development, 82*(4), 1224–1237.

Meck, W. H., & Church, R. M. (1983). A mode control model of counting and timing processes. *Journal of Experimental Psychology. Animal Behavior Processes, 9*(3), 320–334.

Moser, E. I., Kropff, E., & Moser, M.-B. (2008). Place cells, grid cells, and the brain's spatial representation system. *Neuroscience, 31*(1), 69.

Mundy, E., & Gilmore, C. K. (2009). Children's mapping between symbolic and nonsymbolic representations of number. *Journal of Experimental Child Psychology, 103*(4), 490–502.

Nieder, A. (2005). Counting on neurons: The neurobiology of numerical competence. *Nature Reviews. Neuroscience, 6*(3), 177.

Nieder, A. (2012). Supramodal numerosity selectivity of neurons in primate prefrontal and posterior parietal cortices. *Proceedings of the National Academy of Sciences, 109*(29), 11860–11865.

Nieder, A., & Miller, E. K. (2004). Analog numerical representations in rhesus monkeys: Evidence for parallel processing. *Journal of Cognitive Neuroscience, 16*(5), 889–901.

Odic, D., Hock, H., & Halberda, J. (2012). Hysteresis affects approximate number discrimination in young children. *Journal of Experimental Psychology. General, 143*(1), 255–265. http://dx.doi.org/10.1037/a0030825.

Odic, D., Im, H. Y., Eisinger, R., Ly, R., & Halberda, J. (under review). PsiMLE: A maximum-likelihood approach to estimating psychophysical scaling and variability more reliably, efficiently, and flexibly.

Odic, D., Le Corre, M., & Halberda, J. (under review). Children's mappings between number words and the approximate number system.

Odic, D., Libertus, M. E., Feigenson, L., & Halberda, J. (in press). Developmental change in the acuity of approximate number and area representations. *Developmental Psychology*.

Opfer, J. E., & Siegler, R. S. (2007). Representational change and children's numerical estimation. *Cognitive Psychology*, *55*(3), 169–195. http://dx.doi.org/10.1016/j.cogpsych.2006.09.002.

Pailian, H., Libertus, M. E., Feigenson, L., & Halberda, J. (under review). Development of visual working memory in childhood.

Piazza, M., Facoetti, A., Trussardi, A. N., Berteletti, I., Conte, S., Lucangeli, D., et al. (2010). Developmental trajectory of number acuity reveals a severe impairment in developmental dyscalculia. *Cognition*, *116*(1), 33–41. http://dx.doi.org/10.1016/j.cognition.2010.03.012.

Piazza, M., Izard, V., Pinel, P., Le Bihan, D., & Dehaene, S. (2004). Tuning curves for approximate numerosity in the human intraparietal sulcus. *Neuron*, *44*(3), 547–555. http://dx.doi.org/10.1016/j.neuron.2004.10.014.

Pinel, P., Piazza, M., Le Bihan, D., & Dehaene, S. (2004). Distributed and overlapping cerebral representations of number, size, and luminance during comparative judgments. *Neuron*, *41*(6), 983–993.

Price, G. R., Palmer, D., Battista, C., & Ansari, D. (2012). Nonsymbolic numerical magnitude comparison: Reliability and validity of different task variants and outcome measures, and their relationship to arithmetic achievement in adults. *Acta Psychologica*, *140*(1), 50–57. http://dx.doi.org/10.1016/j.actpsy.2012.02.008.

Raposo, D., Sheppard, J. P., Schrater, P. R., & Churchland, A. K. (2012). Multisensory decision-making in rats and humans. *The Journal of Neuroscience*, *32*(11), 3726–3735.

Roitman, J. D., Brannon, E. M., & Platt, M. L. (2007). Monotonic coding of numerosity in macaque lateral intraparietal area. *PLoS Biology*, *5*(8), e208.

Ross, J., & Burr, D. C. (2010). Vision senses number directly. *Journal of Vision*, *10*(2), 10.

Siegler, R. S., & Booth, J. L. (2004). Development of numerical estimation in young children. *Child Development*, *75*(2), 428–444.

Spaepen, E., Coppola, M., Spelke, E. S., Carey, S., & Goldin-Meadow, S. (2011). Number without a language model. *Proceedings of the National Academy of Sciences*, *108*(8), 3163.

Sullivan, J., & Barner, D. (2013). How are number words mapped to approximate magnitudes? *The Quarterly Journal of Experimental Psychology*, *66*(2), 389–402.

Sullivan, J., & Barner, D. (in press). Inference and association in children's early numerical estimation. *Child Development*. doi:10.1111/cdev.12211.

Szucs, D., Nobes, A., Devine, A., Gabriel, F. C., & Gebuis, T. (2013). Visual stimulus parameters seriously compromise the measurement of approximate number system acuity and comparative effects between adults and children. *Developmental Psychology*, *4*, 444. http://dx.doi.org/10.3389/fpsyg.2013.00444.

Wang, J., Odic, D., Halberda, J., & Feigenson, L. (under review). A five-minute intervention improves preschoolers' intuitive sense of number with significant transfer to symbolic mathematics performance.

Whalen, J., Gallistel, C. R., & Gelman, R. (1999). Nonverbal counting in humans: The psychophysics of number representation. *Psychological Science*, *10*(2), 130–137.

Wood, G., Willmes, K., Nuerk, H.-C., & Fischer, M. H. (2008). On the cognitive link between space and number: A meta-analysis of the SNARC effect. *Psychology Science Quarterly*, *50*(4), 489.

Xu, F., & Spelke, E. S. (2000). Large number discrimination in 6-month-old infants. *Cognition*, *74*(1), B1–B11. http://dx.doi.org/10.1016/S0010-0277(99)00066-9.

Xu, F., Spelke, E. S., & Goddard, S. (2005). Number sense in human infants. *Developmental Science*, *8*(1), 88–101.

Zosh, J. M., Halberda, J., & Feigenson, L. (2011). Memory for multiple visual ensembles in infancy. *Journal of Experimental Psychology. General*, *140*(2), 141.